Field theory on kappa-spacetime

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Abstract

A general formalism is developed that allows the construction of field theory on quantum spaces which are deformations of ordinary spacetime. The symmetry group of spacetime is replaced by a quantum group. This formalism is demonstrated for the κ-deformed Poincaré algebra and its quantum space. The algebraic setting is mapped to the algebra of functions of commuting variables with a suitable $\star$-product. Fields are elements of this function algebra. As an example, the Klein-Gordon equation is defined and derived from an action.

I. INTRODUCTION

Although quantum field theory is extremely successful, the combination of general relativity and quantum mechanics suggests that spacetime might not be a differential manifold. Relying on the well-developed mathematical concept of deformation, we formulate a field theory defined on a quantum space rather than on the usual differential manifold [1, 2, 3]. The main idea is the following: A differential manifold can be described by the algebra of functions on the manifold. We deform the usual algebra of functions on Minkowski spacetime to obtain the functions on the $\kappa-$Minkowski spacetime. In constructing the theory, we implement the $\kappa-$Poincare algebra as a deformed symmetry. Physical fields are those functions which are representations of the deformed symmetry algebra. After defining all field-theory ingredients in an abstract algebra formalism, we use the $\star$-product representation to establish a connection with usual field theory. We regard the effective ("noncommutative") action obtained in this way as a smooth deformation of the standard theory, where a small parameter of deformation should be determined by experimental input. For a different interpretation see Ref.[4] and the contributions by N. R. Bruno, and by F. J. Herranz in this Proceedings.

II. ALGEBRAIC SETTING

The $\kappa-$deformed space is the factor space of the algebra freely generated by n coordinates $\hat{x}^1 \ldots \hat{x}^n$, divided by the ideal generated by commutation relations:

$$[\hat{x}^k, \hat{x}^l] = 0,\ [\hat{x}^n, \hat{x}^l] = ia\hat{x}^l, \quad k, l = 1, \ldots n-1. \tag{1}$$

We work in the Euclidean space, where we rotate the deformation vector a^μ into the n-th direction[1].

Derivatives on an algebra have been introduced in [5]. They generate a map in the coordinate space, elements of the coordinate space are mapped to other elements of the coordinate space. Thus, they have to be consistent with the algebra relations and for $a = 0$, they should behave like ordinary derivatives. In addition, they should act at most linearly in the coordinates and the derivatives, and commute among themselves. These requirements are satisfied by the following rules for differentiation[2]:

$$\left[\hat{\partial}_n, \hat{x}^\mu\right] = \delta_n^\mu,\ \left[\hat{\partial}_i, \hat{x}^\mu\right] = \delta_j^\mu + ia\delta_n^\mu\hat{\partial}_i. \tag{2}$$

[1] The deformation parameter $a = a^n$ is related to the more common κ through $\sqrt{a^2} = \kappa^{-1}$.
[2] This solution is not unique, but the ambiguity does not show up in the physical action, see Ref.[3]

Derivatives in the k-th direction have the deformed Leibniz rule:

$$\hat{\partial}_k(\hat{f} \cdot \hat{g}) = (\hat{\partial}_k \hat{f}) \cdot \hat{g} + (e^{ia\hat{\partial}_n} \hat{f}) \cdot \hat{\partial}_k \hat{g}. \tag{3}$$

The symmetry structure of the space is a deformation of the n-dimensional group of rotations. The generators of symmetry are maps of the coordinate space consistent with the relations (1):

$$\begin{aligned} \left[M^{kl}, \hat{x}^\mu\right] &= \delta^{k\mu} \hat{x}^l - \delta^{l\mu} \hat{x}^k, \\ \left[M^{kn}, \hat{x}^\mu\right] &= \delta^{k\mu} \hat{x}^n - \delta^{n\mu} \hat{x}^k - iaM^{k\mu}. \end{aligned} \tag{4}$$

From (4) it is possible to compute the commutators of the generators:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \delta^{\mu\rho} M^{\nu\sigma} + \delta^{\nu\sigma} M^{\mu\rho} - \delta^{\mu\sigma} M^{\nu\rho} - \delta^{\nu\rho} M^{\mu\sigma}. \tag{5}$$

This is the undeformed $SO(n)$ algebra, but the comultiplication is deformed for generators of rotations involving the n-th direction:

$$M^{kn}(\hat{f} \cdot \hat{g}) = \left(M^{kn} \hat{f}\right) \cdot \hat{g} + \left(e^{ia\hat{\partial}_n} \hat{f}\right) \cdot M^{kn} \hat{g} + ia \left(\hat{\partial}_l \hat{f}\right) \cdot M^{kl} \hat{g}. \tag{6}$$

The derivatives introduced in (2) have complicated transformation properties under rotation, see Ref.[1]. For physical applications we construct the derivatives with the usual, undeformed transformation properties under rotation. A deformed Laplace operator (see Refs.[6]) and a deformed Dirac operator (see Refs.[7]) can be defined. For the Laplace operator $\hat{\Box}$, we demand that it should commute with the generators of the algebra $\left[M^{\mu\nu}, \hat{\Box}\right] = 0$, and that it should be a deformation of the usual Laplace operator. By iteration in a we find

$$\hat{\Box} = e^{-ia\hat{\partial}_n} \hat{\Delta} + \frac{2}{a^2}\Big(1 - \cos(a\hat{\partial}_n)\Big). \tag{7}$$

Since the γ-matrices are $\hat{x}$-independent and transform as usual, the covariance of the full Dirac operator $\gamma^\mu \hat{D}_\mu$ implies that the transformation law of its components is vector-like:

$$\left[M^{\mu\nu}, \hat{D}_\rho\right] = \delta^\mu_\rho \hat{D}^\nu - \delta^\nu_\rho \hat{D}^\mu. \tag{8}$$

These relations are obviously consistent with the algebra (5). The components of Dirac operator that satisfy (8) and have the correct limit for $a \to 0$ are

$$\begin{aligned} \hat{D}_n &= \frac{1}{a} \sin(a\hat{\partial}_n) + \frac{ia}{2} \hat{\Delta}\ e^{-ia\hat{\partial}_n}, \\ \hat{D}_i &= \hat{\partial}_i e^{-ia\hat{\partial}_n}. \end{aligned} \tag{9}$$

Physical fields are formal power-series expansions in the coordinates and as such are elements of the coordinate algebra:

$$\hat{\phi}(\hat{x}) = \sum_{\{\alpha\}} c_{\alpha_1 \dots \alpha_n} : (\hat{x}^1)^{\alpha_1} \dots (\hat{x}^n)^{\alpha_n} : . \tag{10}$$

The summation is over a basis in the coordinate algebra, as indicated by colons. The field can also be defined by its coefficient functions $c_{\{\alpha_1 \dots \alpha_n\}}$, once the basis is specified. Fields can be added, multiplied, differentiated and transformed. For example, we define the transformation law of a scalar field as

$$\hat{\phi}(\hat{x}) = (1 + \varepsilon_{\mu\nu} M^{\mu\nu}) \hat{\phi}'(\hat{x}).$$

Because of the nontrivial coproduct of the M^{kn} generator (6), we cannot use the usual $\phi'(x') = \phi(x)$ definition.

Having defined all ingredients, we can write the equation of motion for a free scalar field, invariant under the action of the symmetry generators by construction:

$$\left(\hat{\Box} + m^2\right) \hat{\phi}(\hat{x}) = 0. \tag{11}$$

III. THE $\star$-PRODUCT

The framework of deformation quantization [8], allows to map the associative algebra of functions on noncommutative space to an algebra of functions on a commutative space by means of $\star$-product. In short, the idea is as follows: We consider polynomials of fixed degree in the algebra - homogeneous polynomials. They form a finite-dimensional vector space. For an algebra with the Poincaré-Birkhoff-Witt property (and a Lie algebra has this property), the dimension of the vector space of homogeneous polynomials in the algebra is the same as for polynomials of commuting variables. Thus, there is an isomorphism between two finite-dimensional vector spaces. This vector space isomorphism can be extended to an algebra isomorphism by defining the product of polynomials of commuting variables by first mapping these polynomials back to the algebra, multiplying them there and mapping the product to the space of polynomials of ordinary variables. The product we obtain in this way is called $\star$-product. It is noncommutative and contains the information about the product in the algebra.

An efficient way of computing the $\star$-product is Weyl quantization. Although one can find a closed form for the $\star$-product (see Ref.[3]), it is more instructive to write an expanded expression,

up to second order in deformation parameter a:

$$\begin{aligned} f \star g\,(x) \;=\; & f(x)g(x) + \frac{ia}{2}x^j\Big(\partial_n f(x)\partial_j g(x) - \partial_j f(x)\partial_n g(x)\Big) \\ & -\frac{a^2}{12}x^j\Big(\partial_n^2 f(x)\partial_j g(x) - \partial_j\partial_n f(x)\partial_n g(x) \\ & \qquad -\partial_n f(x)\partial_j\partial_n g(x) + \partial_j f(x)\partial_n^2 g(x)\Big) \\ & -\frac{a^2}{8}x^j x^k\Big(\partial_n^2 f(x)\partial_j\partial_k g(x) - 2\partial_j\partial_n f(x)\partial_n\partial_k g(x) \\ & \qquad +\partial_j\partial_k f(x)\partial_n^2 g(x)\Big) + \mathcal{O}(a^3). \end{aligned} \tag{12}$$

From the action of an operator $\hat{O}$ on symmetric polynomials in the algebra we compute the action of an operator O^* on ordinary functions. For example,

$$\begin{aligned} \partial_i^* f(x) &= \partial_i \frac{e^{ia\partial_n}-1}{ia\partial_n} f(x), \\ M^{*ln} f(x) &= \left(x^l\partial_n - x^n\partial_l + x^l\partial_\mu\partial_\mu \frac{e^{ia\partial_n}-1}{2\partial_n} - x^\nu\partial_\nu\partial_l \frac{e^{ia\partial_n}-1-ia\partial_n}{ia\partial_n^2}\right) f(x). \end{aligned}$$

The operators inherit the Leibniz rule from the algebra.

Now we can write the Klein-Gordon equation of motion (11) in the following form:

$$\left(\Box^* + m^2\right)\phi(x) = \left(-\frac{2}{a^2\partial_n^2}(\cos(a\partial_n)-1)\Box + m^2\right)\phi(x) = 0. \tag{13}$$

Expanding the equation in a will give us the equation of motion for the free scalar field with second-order correction:

$$\left(\Box + m^2 - \frac{a^2\partial_n^2}{12}\Box + \mathcal{O}(a^3)\right)\phi(x) = 0. \tag{14}$$

IV. THE VARIATIONAL PRINCIPLE

We derive field equations by means of a variational principle such that the dynamics can be formulated with the help of the Lagrangian formalism. For this purpose, we need an integral. We define it in the $\star$-product formalism and use the usual definition of an integral of functions of commuting variables. Such an integral in general will not have the trace property, but we can introduce a measure function to achieve it:

$$\int \mathrm{d}^n x\; \mu(x)\; (f(x)\star g(x)) = \int \mathrm{d}^n x\; \mu(x)\; (g(x)\star f(x)). \tag{15}$$

Note that $\mu(x)$ is not $\star$-multiplied with the other functions, it is part of the volume element. Using Eq.(15) as a definition of the measure function, and Eq.(12), we obtain

$$\partial_n\mu(x) = 0, \qquad x^j\partial_j\mu(x) = (1-n)\mu(x).$$

We also have to define "improved" differential operators $\mathcal{O}$ which are hermitean in the sense

$$\int \mathrm{d}^n x \; \mu \; \bar{f} \star \mathcal{O} g = \int \mathrm{d}^n x \; \mu \; \overline{\mathcal{O} f} \star g. \tag{16}$$

This is achieved by the following redefinition of all differential operators:

$$\partial_k^* \to \tilde{\partial}_k^* = \left(\partial_k + \frac{\partial_k \mu}{2\mu} \right) \frac{e^{ia\partial_n} - 1}{ia\partial_n}. \tag{17}$$

Now we define the variational principle in such a way that the function to be varied is brought to the left by cyclic permutation and varied:

$$\frac{\delta}{\delta g} \int \mathrm{d}^n x \; \mu \; f \star g \star h = \frac{\delta}{\delta g} \int \mathrm{d}^n x \; \mu \; g(h \star f) = \mu \; (h \star f). \tag{18}$$

With this definition, after performing a suitable field redefinition [1], we derive the equation of motion (13) from the following action:

$$S = \frac{1}{2} \int \mathrm{d}^n x \mu \; \phi(x) \star \left(\tilde{\Box}^* + m^2 \right) \phi(x). \tag{19}$$

The operator $\tilde{\Box}^*$ is the improved Laplace operator $\Box^*$ in the sense of (17).

V. OUTLOOK

Using the formalism developed in Ref.[1] and presented here, one can also construct gauge theories on κ-spacetime, see Ref.[2]. The main consequences of deformation of the coordinate algebra for gauge theories are:

a) Gauge fields are enveloping-algebra-valued, and, therefore, one must construct a (Seiberg-Witten) map to restrict the theory to the finite (Lie-algebra) number of degrees of freedom.

b) Gauge fields are derivative-valued, as a consequence of deformed Leibniz rules.

An important open problem is the construction of an invariant action. Namely, we can construct an action invariant under gauge transformation OR an action invariant under the action of symmetry generators (using the "quantum trace" instead of the integral defined in (15), see Ref.[9]). The work on the problems of quantization and on the formulation of "deformed" conservation laws is in progress.

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